Chapter 8:
Generalization and Function Approximation

Objectives of this chapter:

- Look at how experience with a limited part of the state set be used to produce good behavior over a much larger part.
- Overview of function approximation (FA) methods and how they can be adapted to RL

Value Prediction with FA

As usual: Policy Evaluation (the prediction problem):

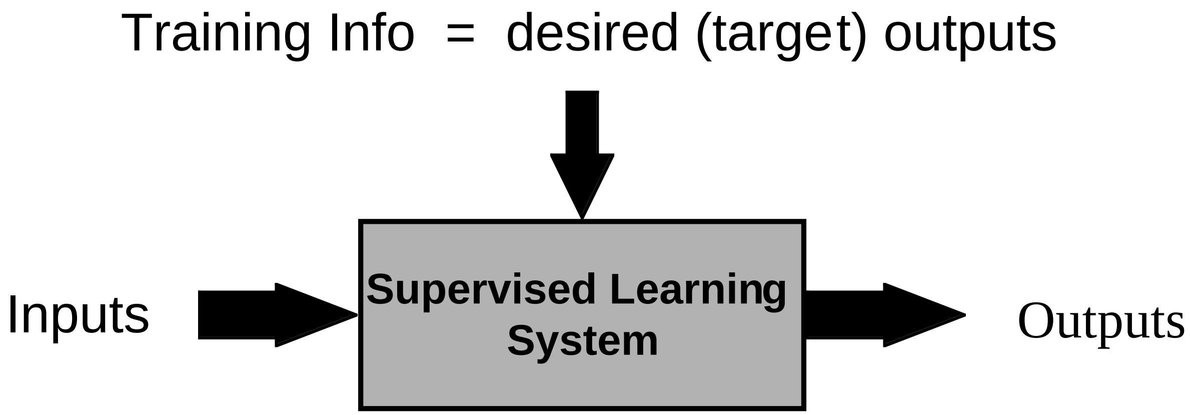
for a given policy π, compute the state-value function V^{π}

In earlier chapters, value functions were stored in lookup tables.

Here, the value function estimate at time t, V_t, depends on a **parameter vector** $\vec{\theta}_t$, and only the parameter vector is updated.

e.g., $\vec{\theta}_t$ could be the vector of connection weights of a neural network.

Adapt Supervised Learning Algorithms



Training Info = desired (target) outputs

Inputs

Supervised Learning System

Outputs

Training example = {input, target output}

Error = (target output – actual output)

Backups as Training Examples

e.g., the TD(0) backup:

$$V(s_t) \leftarrow V(s_t) + \alpha\left[r_{t+1} + \gamma V(s_{t+1}) - V(s_t)\right]$$

As a training example:

$$\left\{\text{description of } s_t,\ \ r_{t+1} + \gamma V(s_{t+1})\right\}$$

input

target output

Any FA Method?

- In principle, yes:
  - artificial neural networks
  - decision trees
  - multivariate regression methods
  - etc.
- But RL has some special requirements:
  - usually want to learn while interacting
  - ability to handle nonstationarity
  - other?

Gradient Descent Methods

$$\vec{\theta}_{\mathrm{t}} = \left(\theta_t(1), \theta_t(2), \ldots, \theta_t(n)\right)^T$$

transpose

Assume V_t is a (sufficiently smooth) differentiable function of $\vec{\theta}_t$, for all $s \in S$.

Assume, for now, training examples of this form:

$$\left\{\text{description of } s_t,\ V^\pi(s_t)\right\}$$

Performance Measures

- Many are applicable but…
- a common and simple one is the mean-squared error (MSE) over a distribution P :

$$MSE(\theta_t) = \sum_{s \in S} P(s)\left[V^{\pi}(s) - V_t(s)\right]^2$$

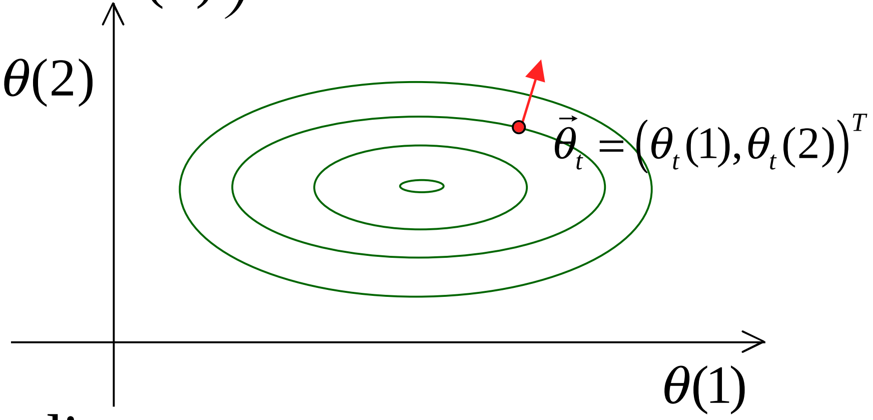
- Why P ?
- Why minimize MSE?
- Let us assume that P is always the distribution of states at which backups are done.
- The **on-policy distribution**: the distribution created while following the policy being evaluated. Stronger results are available for this distribution.

Gradient Descent

Let f be any function of the parameter space.

Its gradient at any point $\vec{\theta}_t$ in this space is:

$$\nabla_{\vec{\theta}} f(\vec{\theta}_t) = \left(\frac{\partial f(\vec{\theta}_t)}{\partial \theta(1)}, \frac{\partial f(\vec{\theta}_t)}{\partial \theta(2)}, \ldots, \frac{\partial f(\vec{\theta}_t)}{\partial \theta(n)} \right)^T .$$



Iteratively move down the gradient:

$$\vec{\theta}_{t+1} = \vec{\theta}_t - \alpha \nabla_{\vec{\theta}} f(\vec{\theta}_t)$$

Gradient Descent Cont.

For the MSE given above and using the chain rule:

$$\begin{aligned}
\vec{\theta}_{t+1} &= \vec{\theta}_t - \frac{1}{2}\alpha\nabla_{\vec{\theta}} MSE(\vec{\theta}_t) \\
&= \vec{\theta}_t - \frac{1}{2}\alpha\nabla_{\vec{\theta}} \sum_{s\in S} P(s)\left[V^{\pi}(s) - V_t(s)\right]^2 \\
&= \vec{\theta}_t + \alpha \sum_{s\in S} P(s)\left[V^{\pi}(s) - V_t(s)\right]\nabla_{\vec{\theta}} V_t(s)
\end{aligned}$$

Gradient Descent Cont.

Use just the **sample gradient** instead:

$$\begin{aligned}\vec{\theta}_{t+1} &= \vec{\theta}_t - \frac{1}{2}\alpha\nabla_{\vec{\theta}}\left[V^{\pi}(s_t) - V_t(s_t)\right]^2 \\ &= \vec{\theta}_t + \alpha\left[V^{\pi}(s_t) - V_t(s_t)\right]\nabla_{\vec{\theta}}V_t(s_t),\end{aligned}$$

Since each sample gradient is an **unbiased estimate** of the true gradient, this converges to a local minimum of the MSE if α decreases appropriately with *t*.

$$E\left[V^{\pi}(s_t) - V_t(s_t)\right]\nabla_{\vec{\theta}}V_t(s_t) = \sum_{s\in S} P(s)\left[V^{\pi}(s) - V_t(s)\right]\nabla_{\vec{\theta}}V_t(s)$$

But We Don't have these Targets

Suppose we just have targets v_t instead:

$$\vec{\theta}_{t+1} = \vec{\theta}_t + \alpha\left[v_t - V_t(s_t)\right]\nabla_{\vec{\theta}} V_t(s_t)$$

If each v_t is an unbiased estimate of $V^\pi(s_t)$,
i.e., $E\{v_t\} = V^\pi(s_t)$, then gradient descent converges
to a local minimum (provided α decreases appropriately).

e.g., the Monte Carlo target $v_t = R_t$:

$$\vec{\theta}_{t+1} = \vec{\theta}_t + \alpha\left[R_t - V_t(s_t)\right]\nabla_{\vec{\theta}} V_t(s_t)$$

What about TD(λ) Targets?

$$\vec{\theta}_{t+1} = \vec{\theta}_t + \alpha\left[R_t^\lambda - V_t(s_t)\right]\nabla_{\vec{\theta}} V_t(s_t)$$

Not for $\lambda < 1$

But we do it anyway, using the backwards view:

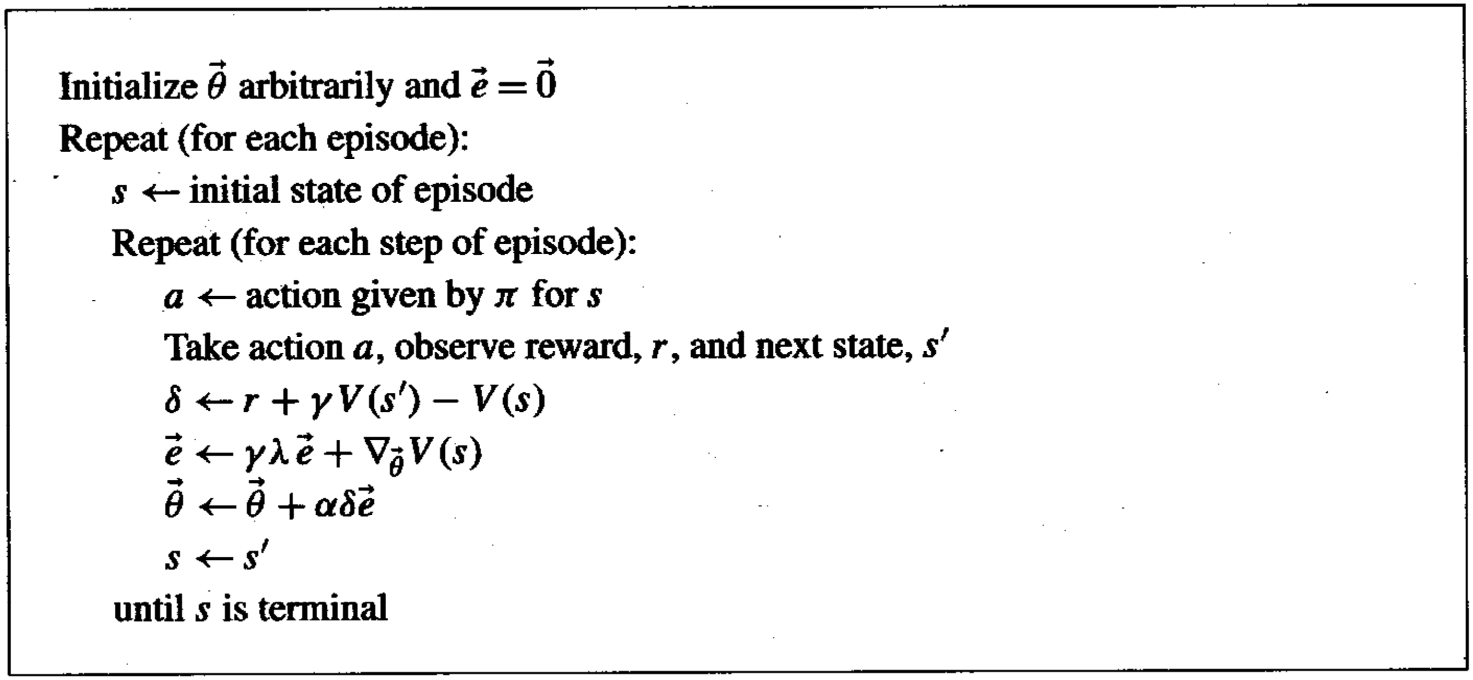
$$\vec{\theta}_{t+1} = \vec{\theta}_t + \alpha\,\delta_t\,\vec{e}_t,$$

where:

$\delta_t = r_{t+1} + \gamma V_t(s_{t+1}) - V_t(s_t)$, as usual, and

$$\vec{e}_t = \gamma\lambda\vec{e}_{t-1} + \nabla_{\vec{\theta}} V_t(s_t)$$

On-Line Gradient-Descent TD(λ)



```
Initialize $\vec{\theta}$ arbitrarily and $\vec{e} = \vec{0}$
Repeat (for each episode):
    $s \leftarrow$ initial state of episode
    Repeat (for each step of episode):
        $a \leftarrow$ action given by $\pi$ for $s$
        Take action $a$, observe reward, $r$, and next state, $s'$
        $\delta \leftarrow r + \gamma V(s') - V(s)$
        $\vec{e} \leftarrow \gamma\lambda\vec{e} + \nabla_{\vec{\theta}} V(s)$
        $\vec{\theta} \leftarrow \vec{\theta} + \alpha\delta\vec{e}$
        $s \leftarrow s'$
    until $s$ is terminal
```

Linear Methods

Represent states as feature vectors:

for each $s \in S$:

$$\vec{\phi}_s = \left(\phi_s(1), \phi_s(2), \ldots, \phi_s(n)\right)^T$$

$$V_t(s) = \vec{\theta}_t^{\,T} \vec{\phi}_s = \sum_{i=1}^{n} \theta_t(i) \phi_s(i)$$

$$\nabla_{\vec{\theta}} V_t(s) = \quad ?$$

Nice Properties of Linear FA Methods

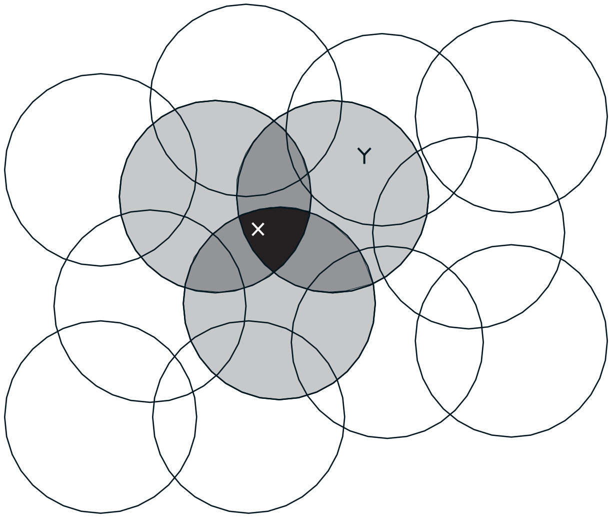
- The gradient is very simple: $\nabla_{\vec{\theta}} V_t(s) = \vec{\phi}_s$
- For MSE, the error surface is simple: quadratic surface with a single minumum.
- Linear gradient descent TD(λ) converges:
  - Step size decreases appropriately
  - On-line sampling (states sampled from the on-policy distribution)
  - Converges to parameter vector $\vec{\theta}_\infty$ with property:

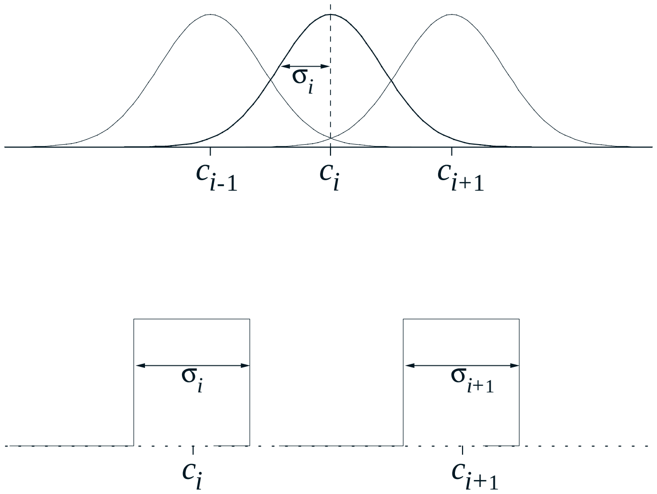
$$MSE(\vec{\theta}_\infty) \leq \frac{1-\gamma\lambda}{1-\gamma} MSE(\vec{\theta}^*)$$

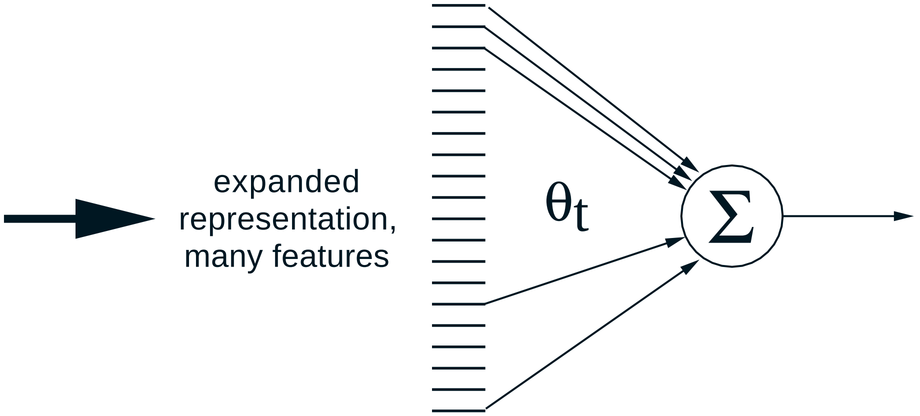
best parameter vector

(Tsitsiklis & Van Roy, 1997)

Coarse Coding





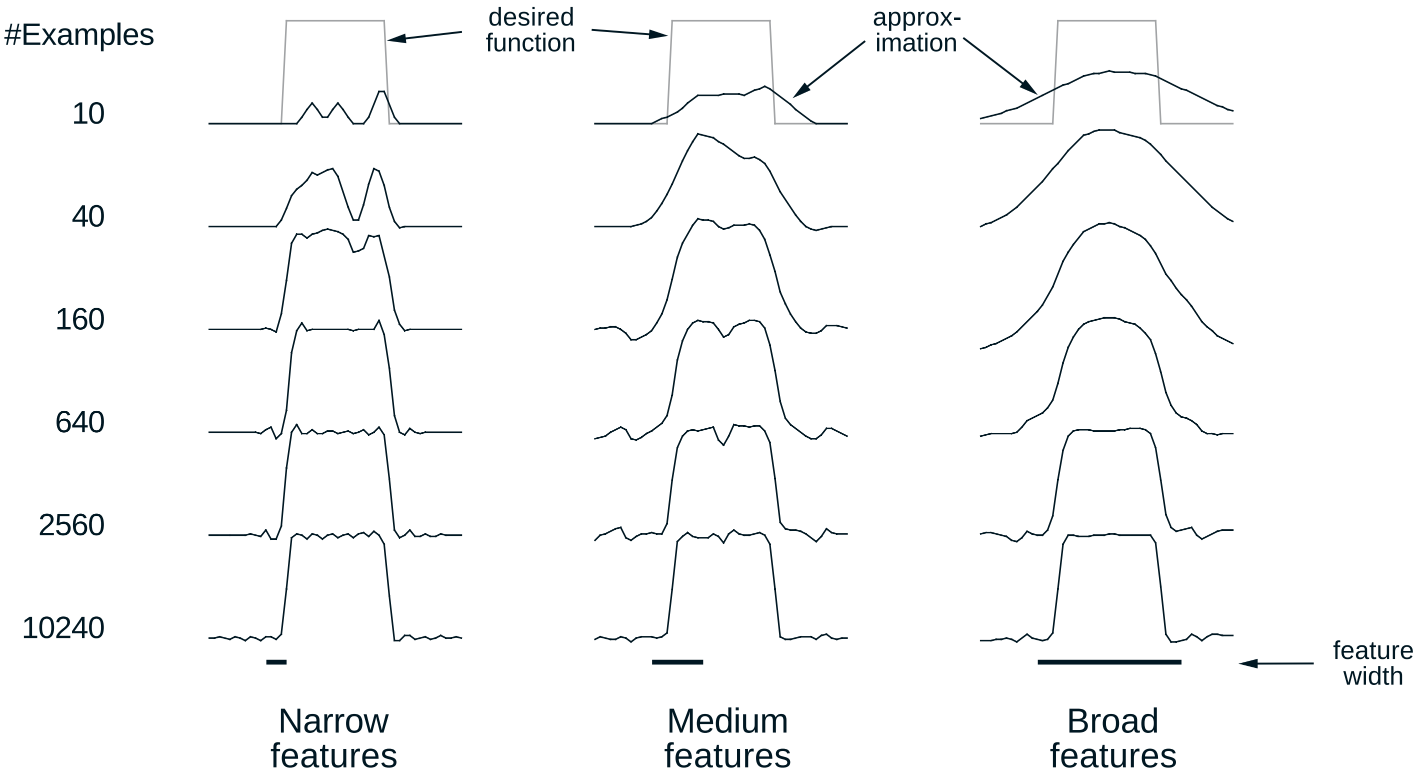


original representation

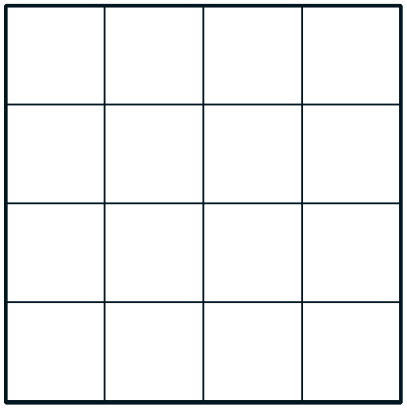
expanded representation, many features

approximation

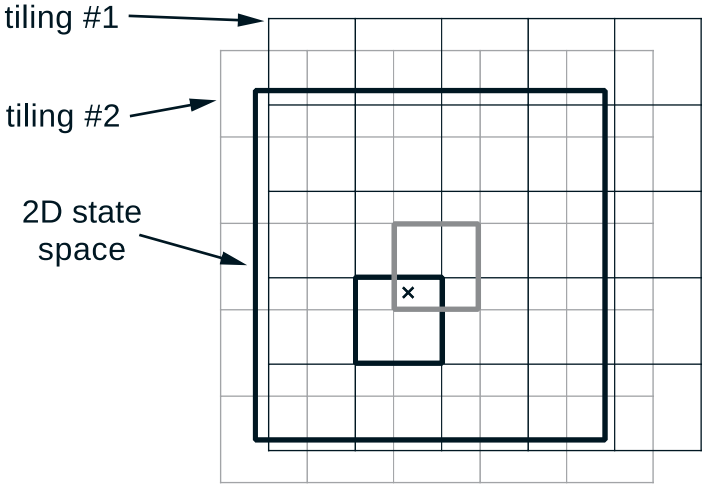
Learning and Coarse Coding



Tile Coding



- Binary feature for each tile
- Number of features present at any one time is constant
- Binary features means weighted sum easy to compute
- Easy to compute indices of the freatures present

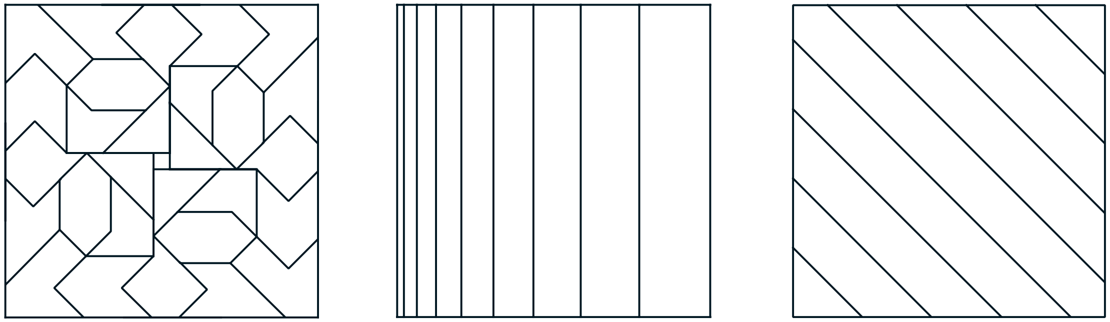


Shape of tiles ⇒ Generalization

#Tilings ⇒ Resolution of final approximation

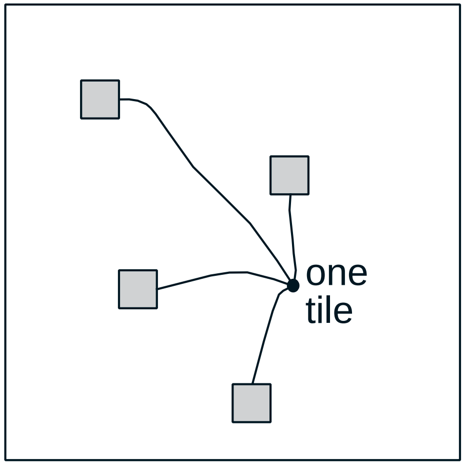
Tile Coding Cont.

Irregular tilings



a) Irregular b) Log stripes c) Diagonal stripes

Hashing

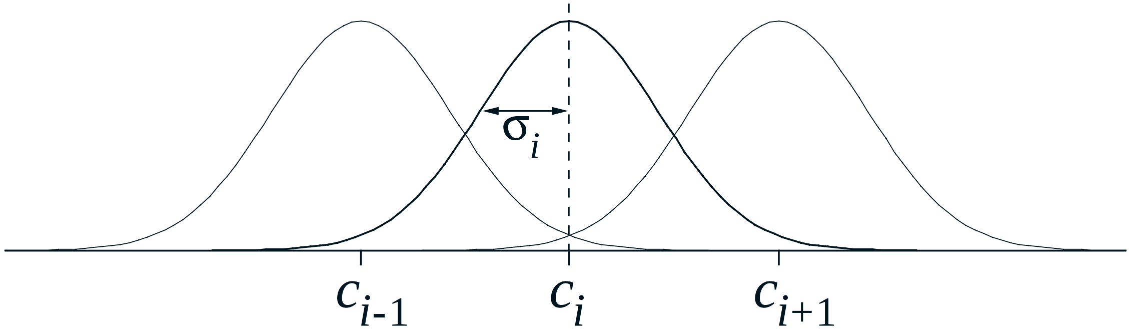


CMAC
 “Cerebellar model arithmetic computer”
Albus 1971

Radial Basis Functions (RBFs)

e.g., Gaussians

$$\phi_s(i) = \exp\left(-\frac{\left\|s - c_i\right\|^2}{2\sigma_i^2}\right)$$



Can you beat the "curse of dimensionality"?

- Can you keep the number of features from going up exponentially with the dimension?
- Function complexity, not dimensionality, is the problem.
- Kanerva coding:
  - Select a bunch of binary **prototypes**
  - Use hamming distance as distance measure
  - Dimensionality is no longer a problem, only complexity
- "Lazy learning" schemes:
  - Remember all the data
  - To get new value, find nearest neighbors and interpolate
  - e.g., locally-weighted regression

Control with FA

- ❐ Learning state-action values

  Training examples of the form:

$$\left\{\text{description of } (s_t, a_t),\ \ v_\mathrm{t}\right\}$$

- ❐ The general gradient-descent rule:

$$\vec{\theta}_{t+1} = \vec{\theta}_t + \alpha\left[v_t - Q_t(s_t, a_t)\right]\nabla_{\vec{\theta}} Q(s_t, a_t)$$

- ❐ Gradient-descent Sarsa(λ) (backward view):

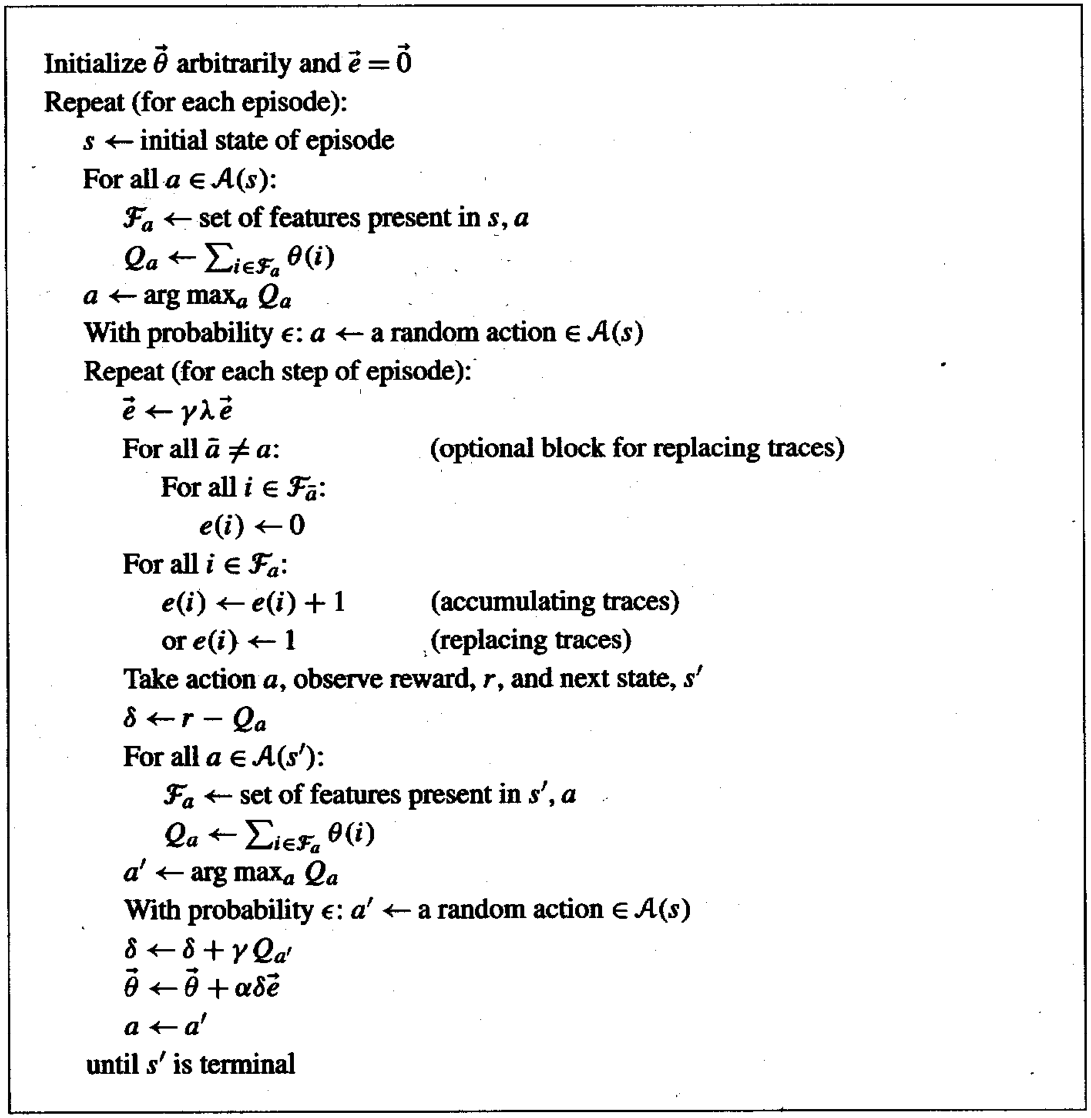
$$\vec{\theta}_{t+1} = \vec{\theta}_t + \alpha\,\delta_t\,\vec{e}_t$$

where

$$\delta_t = r_{t+1} + \gamma\,Q_t(s_{t+1}, a_{t+1}) - Q_t(s_t, a_t)$$

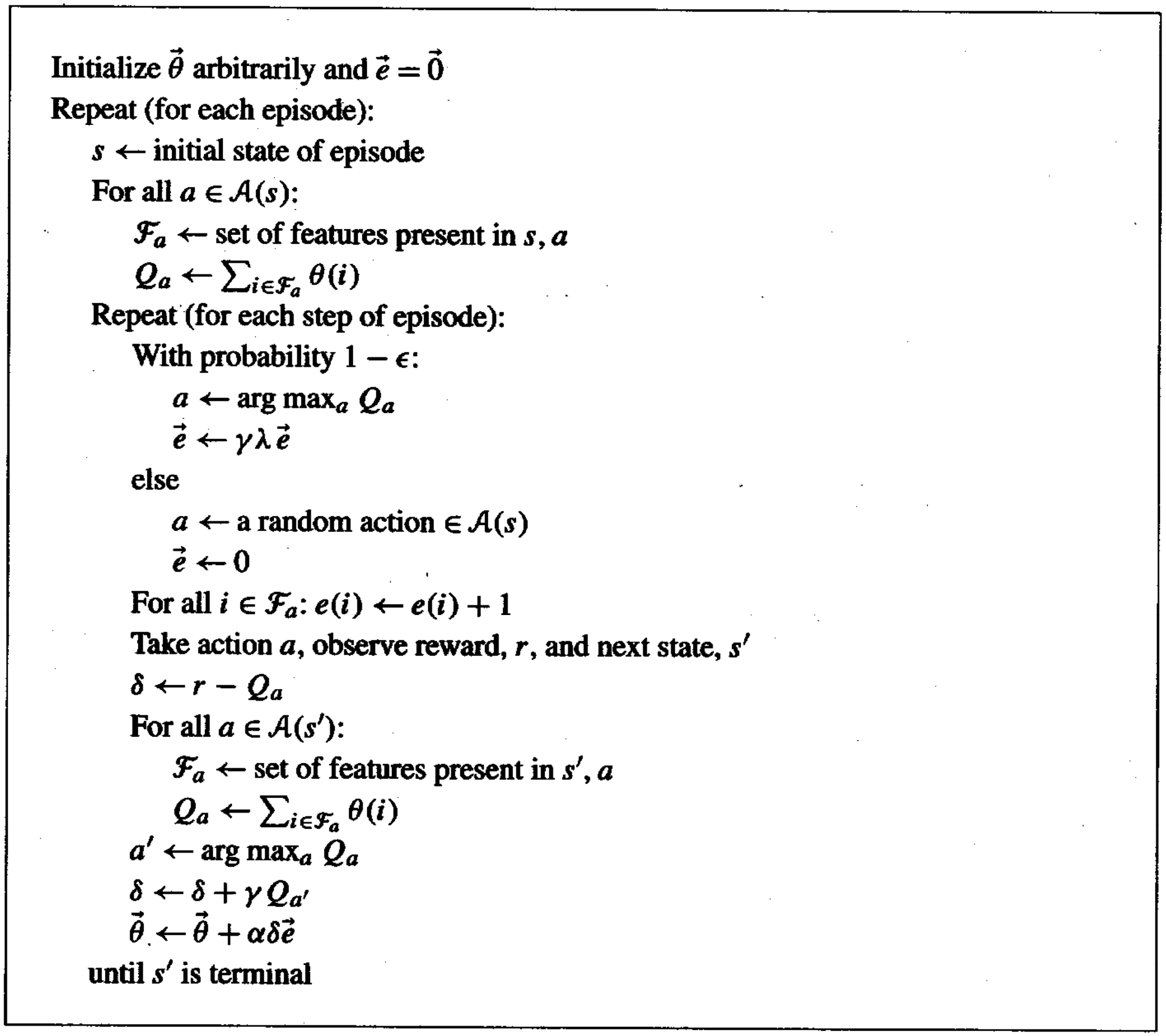
$$\vec{e}_t = \gamma\,\lambda\,\vec{e}_{t-1} + \nabla_{\vec{\theta}}\vec{Q}_t(s_t, a_t)$$

GPI with Linear Gradient Descent Sarsa(λ)



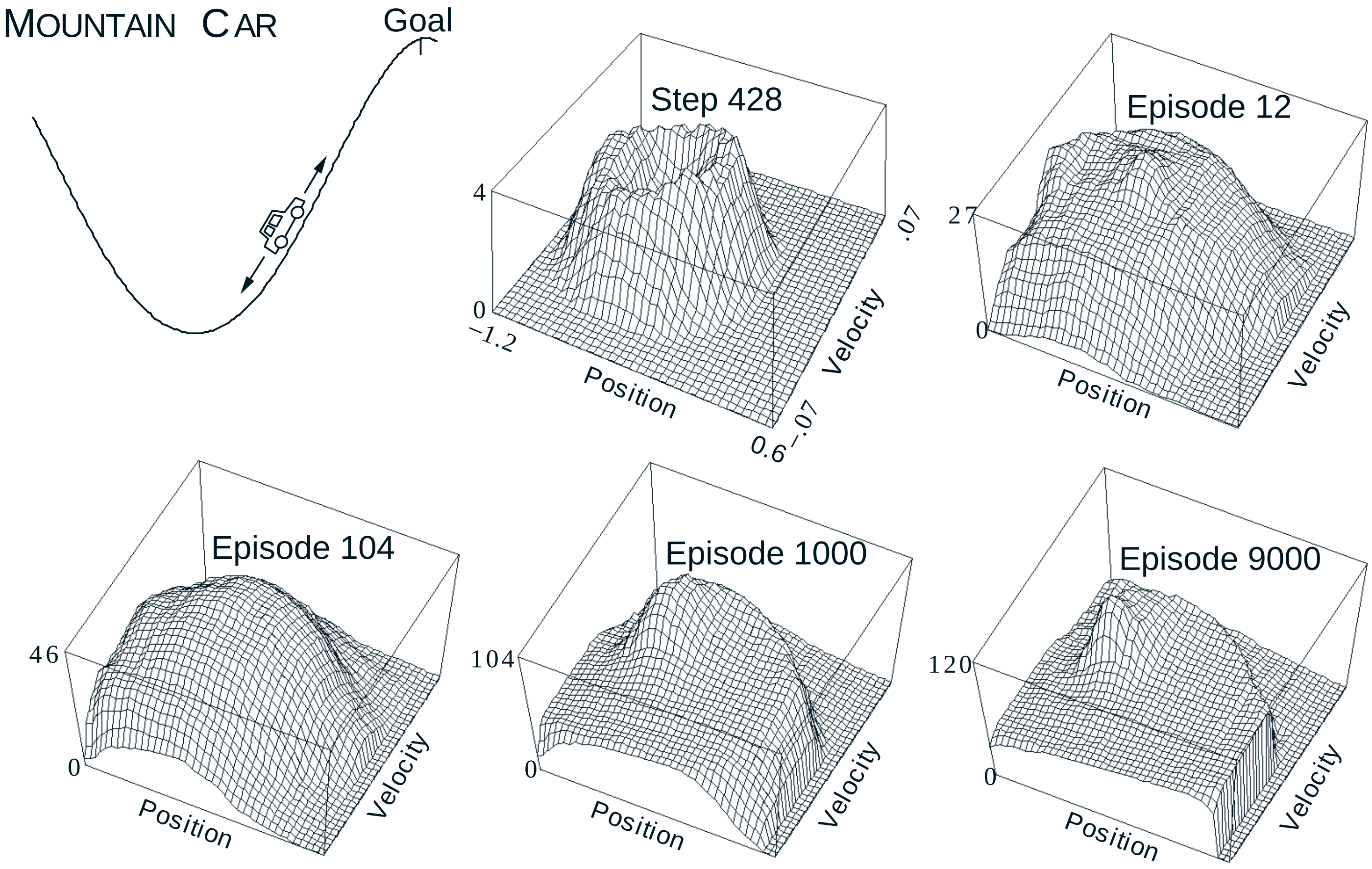
```
Initialize $\vec{\theta}$ arbitrarily and $\vec{e} = \vec{0}$
Repeat (for each episode):
    $s \leftarrow$ initial state of episode
    For all $a \in \mathcal{A}(s)$:
        $\mathcal{F}_a \leftarrow$ set of features present in $s, a$
        $Q_a \leftarrow \sum_{i \in \mathcal{F}_a} \theta(i)$
    $a \leftarrow \arg\max_a Q_a$
    With probability $\epsilon$: $a \leftarrow$ a random action $\in \mathcal{A}(s)$
    Repeat (for each step of episode):
        $\vec{e} \leftarrow \gamma\lambda\vec{e}$
        For all $\bar{a} \neq a$:                  (optional block for replacing traces)
            For all $i \in \mathcal{F}_{\bar{a}}$:
                $e(i) \leftarrow 0$
        For all $i \in \mathcal{F}_a$:
            $e(i) \leftarrow e(i) + 1$             (accumulating traces)
            or $e(i) \leftarrow 1$                 (replacing traces)
        Take action $a$, observe reward, $r$, and next state, $s'$
        $\delta \leftarrow r - Q_a$
        For all $a \in \mathcal{A}(s')$:
            $\mathcal{F}_a \leftarrow$ set of features present in $s', a$
            $Q_a \leftarrow \sum_{i \in \mathcal{F}_a} \theta(i)$
        $a' \leftarrow \arg\max_a Q_a$
        With probability $\epsilon$: $a' \leftarrow$ a random action $\in \mathcal{A}(s)$
        $\delta \leftarrow \delta + \gamma Q_{a'}$
        $\vec{\theta} \leftarrow \vec{\theta} + \alpha\delta\vec{e}$
        $a \leftarrow a'$
    until $s'$ is terminal
```

GPI Linear Gradient Descent Watkins' Q(λ)



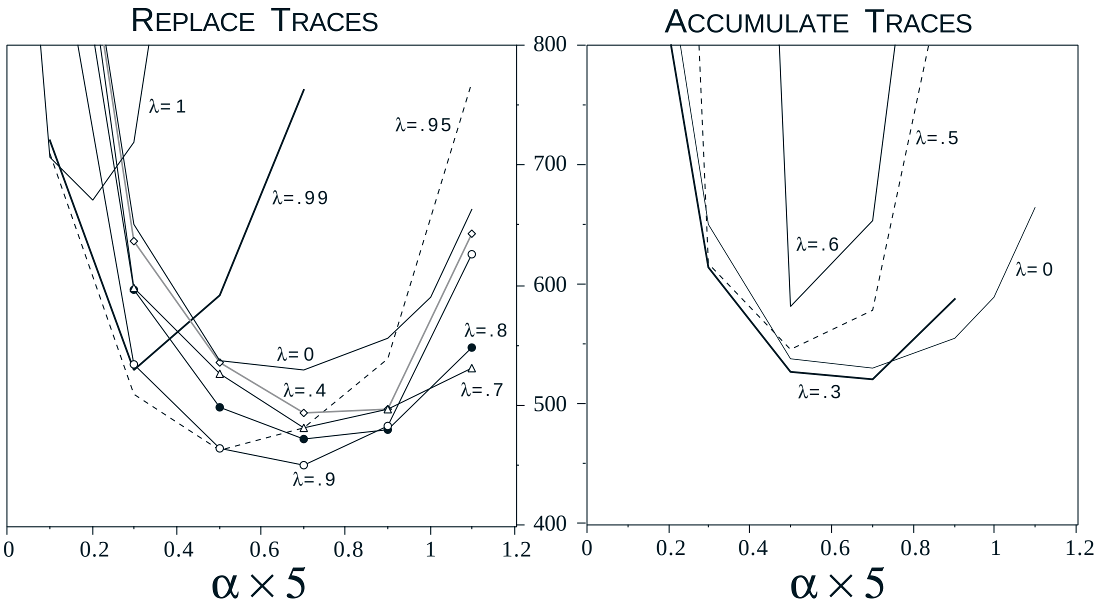
```
Initialize $\vec{\theta}$ arbitrarily and $\vec{e} = \vec{0}$
Repeat (for each episode):
    $s \leftarrow$ initial state of episode
    For all $a \in \mathcal{A}(s)$:
        $\mathcal{F}_a \leftarrow$ set of features present in $s, a$
        $Q_a \leftarrow \sum_{i \in \mathcal{F}_a} \theta(i)$
    Repeat (for each step of episode):
        With probability $1 - \epsilon$:
            $a \leftarrow \arg\max_a Q_a$
            $\vec{e} \leftarrow \gamma\lambda\vec{e}$
        else
            $a \leftarrow$ a random action $\in \mathcal{A}(s)$
            $\vec{e} \leftarrow 0$
        For all $i \in \mathcal{F}_a$: $e(i) \leftarrow e(i) + 1$
        Take action $a$, observe reward, $r$, and next state, $s'$
        $\delta \leftarrow r - Q_a$
        For all $a \in \mathcal{A}(s')$:
            $\mathcal{F}_a \leftarrow$ set of features present in $s', a$
            $Q_a \leftarrow \sum_{i \in \mathcal{F}_a} \theta(i)$
        $a' \leftarrow \arg\max_a Q_a$
        $\delta \leftarrow \delta + \gamma Q_{a'}$
        $\vec{\theta} \leftarrow \vec{\theta} + \alpha\delta\vec{e}$
    until $s'$ is terminal
```

Mountain-Car Task

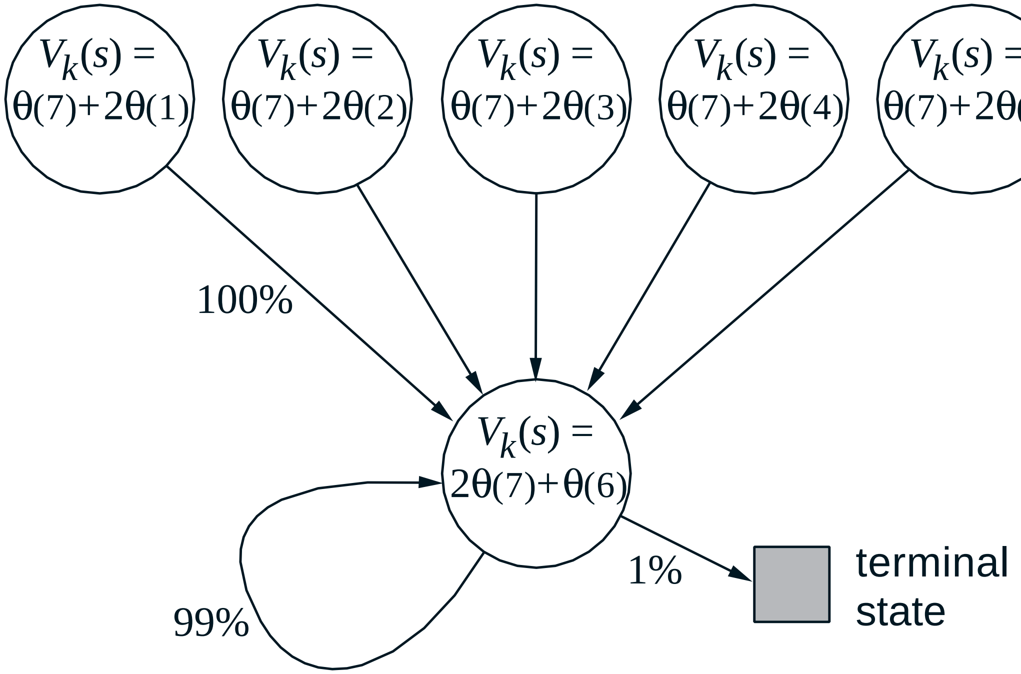


Mountain-Car Results

Steps per episode
averaged over
first 20 trials
and 30 runs

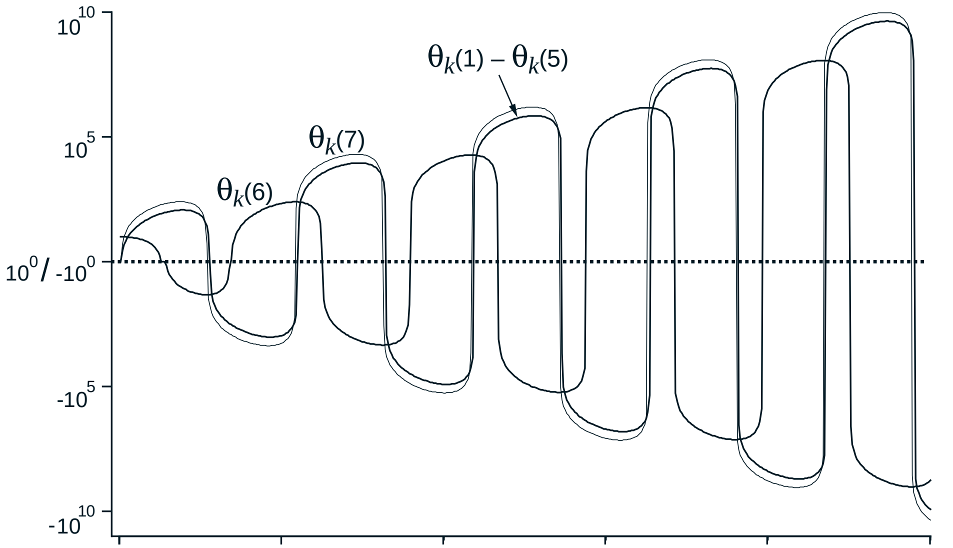


Baird's Counterexample


$V_k(s) = \theta(7)+2\theta(1)$
$V_k(s) = \theta(7)+2\theta(2)$
$V_k(s) = \theta(7)+2\theta(3)$
$V_k(s) = \theta(7)+2\theta(4)$
$V_k(s) = \theta(7)+2\theta(5)$
100%
$V_k(s) = 2\theta(7)+\theta(6)$
99%
1%
terminal state

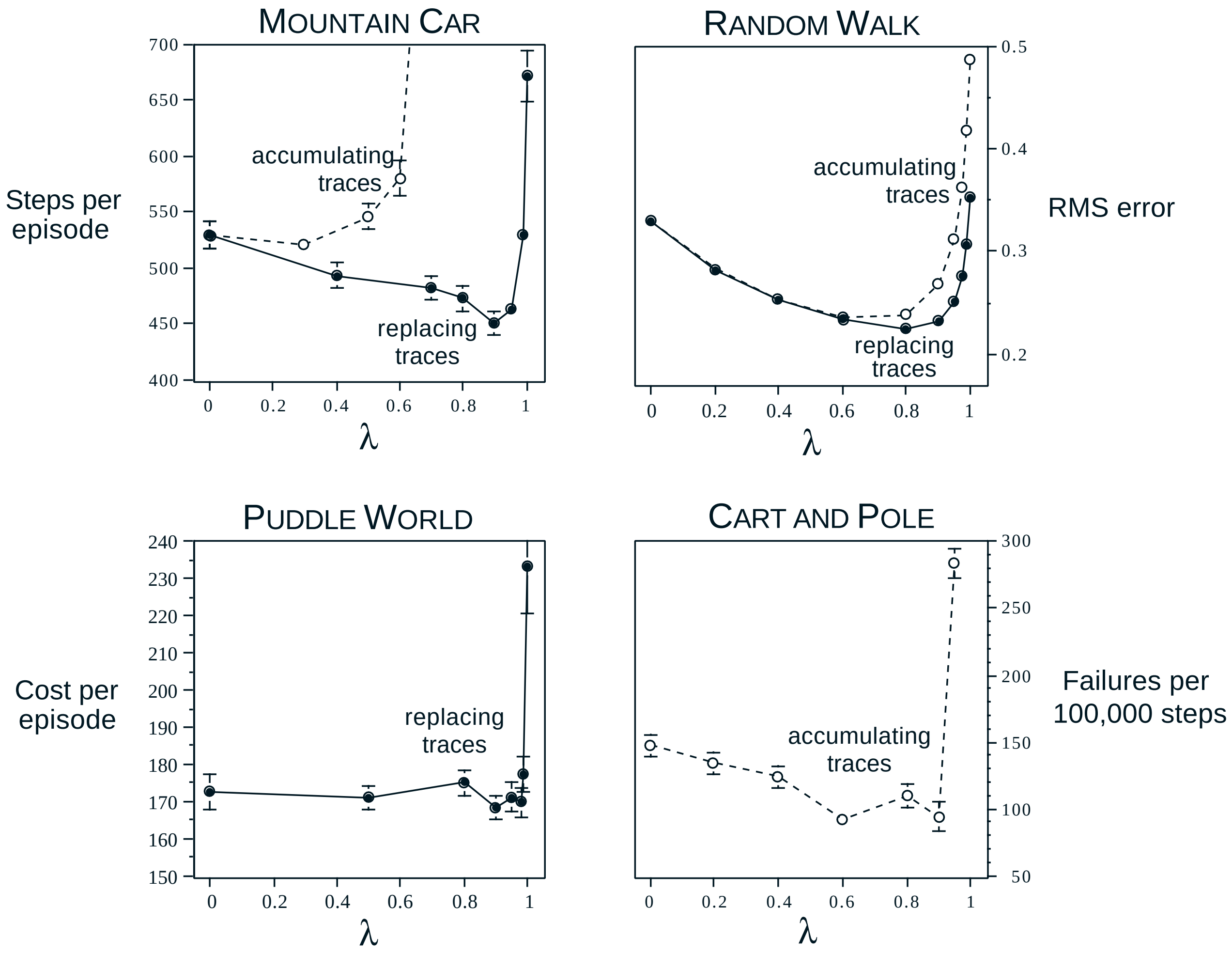
Baird’s Counterexample Cont.

Parameter values, $\theta_k(i)$ (log scale, broken at ±1)



Iterations (k)

Should We Bootstrap?



Summary

- Generalization
- Adapting supervised-learning function approximation methods
- Gradient-descent methods
- Linear gradient-descent methods
  - Radial basis functions
  - Tile coding
  - Kanerva coding
- Nonlinear gradient-descent methods? Backpropation?
- Subleties involving function approximation, bootstrapping and the on-policy/off-policy distinction