

Estimating quantiles

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The p th quantile is defined as the value where the estimated cumulative distribution function is equal to p . As with quantiles in unweighted data, this definition only pins down the quantile to an interval between two observations, and a rule is needed to interpolate. As the help for the base R function `quantile` explains, even before considering sampling weights there are many possible rules.

Rules in the `svyquantile()` function can be divided into three classes

- Discrete rules, following types 1 to 3 in `quantile`
- Continuous rules, following types 4 to 9 in `quantile`
- A rule proposed by Shah & Vaish (2006) and used in some versions of SUDAAN

Discrete rules

These are based on the discrete empirical CDF that puts weight proportional to the weight w_k on values x_k .

$$\hat{F}(x) = \frac{\sum_i \{x_i \leq x\} w_i}{\sum_i w_i}$$

The mathematical inverse The mathematical inverse $\hat{F}^{-1}(p)$ of the CDF is the smallest x such that $F(x) \geq p$. This is rule `hf1` and `math` and in equally-weighted data gives the same answer as `type=1` in `quantile`

The primary-school median The school definition of the median for an even number of observations is the average of the middle two observations. We extend this to say that the p th quantile is $q_{\text{low}} = \hat{F}^{-1}(p)$ if $\hat{F}(q_{\text{low}}) = p$ and otherwise is the the average of $\hat{F}^{-1}(p)$ and the next higher observation. This is `school` and `hf2` and is the same as `type=2` in `quantile`.

Nearest even order statistic The p th quantile is whichever of $\hat{F}^{-1}(p)$ and the next higher observation is at an even-numbered position when the distinct data values are sorted. This is `hf3` and is the same as `type=3` in `quantile`.

Continuous rules

These construct the empirical CDF as a piecewise-linear function and read off the quantile. They differ in the choice of points to interpolate. Hyndman & Fan describe these as interpolating the points (p_k, x_k) where p_k is defined in terms of k and n . For weighted use they have been redefined in terms of the cumulative weights $C_k = \sum_{i \leq k} w_i$, the total weight $C_n = \sum w_i$, and the weight w_k on the k th observation.

| qrule | Hyndman & Fan | Weighted |
|-------|-----------------------------|--------------------------------------|
| hf4 | $p_k = k/n$ | $p_k = C_k/C_n$ |
| hf5 | $p_k = (k - 0.5)/n$ | $p_k = (C_k - w_k)/C_n$ |
| hf6 | $p_k = k/(n + 1)$ | $p_k = C_k/(C_n + w_n)$ |
| hf7 | $p_k = (k - 1)/(n - 1)$ | $p_k = C_{k-1}/C_{n-1}$ |
| hf8 | $p_k = (k - 1/3)/(n + 2/3)$ | $p_k = (C_k - w_k/3)/(C_n + w_n/3)$ |
| hf9 | $p_k = (k - 3/8)/(n + 1/4)$ | $p_k = (C_k - 3w_k/8)/(C_n + w_n/4)$ |

Shah & Vaish

This rule is related to hf6, but it is discrete and more complicated. First, define $w_i^* = w_i n / C_n$, so that w_i^* sum to the sample size rather than the population size, and C^*k as partial sums of w_k^* . Now define the estimated CDF by

$$\hat{F}(x_k) = \frac{1}{n+1} (C_k^* + 1/2 - w_k/2)$$

and take $\hat{F}^{-1}(p)$ as the p th quantile.

Other options

It would be possible to redefine all the continuous estimators in terms of w^* , so that type 8, for example, would use

$$p_k = (C_k^* - 1/3)/(C_n^* + 2/3)$$

Or a compromise, eg using w_k^* in the numerator and numbers in the denominator, such as

$$p_k = (C_k^* - w_k^*/3)/(C_n^* + 2/3).$$

Comparing these would be ~~a worthwhile~~... an interesting... a research question for simulation.